CHAPTER 11

STRAINS BEYOND THE ELASTIC LIMIT

11.1. Introduction

When the design of components is based upon the elastic theory, e.g. the simple bending or torsion theory, the dimensions of the components are arranged so that the maximum stresses which are likely to occur under service loading conditions do not exceed the allowable working stress for the material in either tension or compression. The allowable working stress is taken to be the yield stress of the material divided by a convenient safety factor (usually based on design codes or past experience) to account for unexpected increase in the level of service loads. If the maximum stress in the component is likely to exceed the allowable working stress, the component is considered unsafe, yet it is evident that complete failure of the component is unlikely to occur even if the yield stress is reached at the outer fibers provided that some portion of the component remains elastic and capable of carrying load, i.e. the strength of a component will normally be much greater than that assumed on the basis of initial yielding at any position. To take advantage of the inherent additional strength, therefore, a different design procedure is used which is often referred to as **plastic limit design**. The revised design procedures are based upon a number of basic assumptions about the material behavior. Figure 11.1 shows a typical stress-strain curve for annealed low carbon steel indicating the presence of both upper and lower yield points and strain-hardening characteristics.

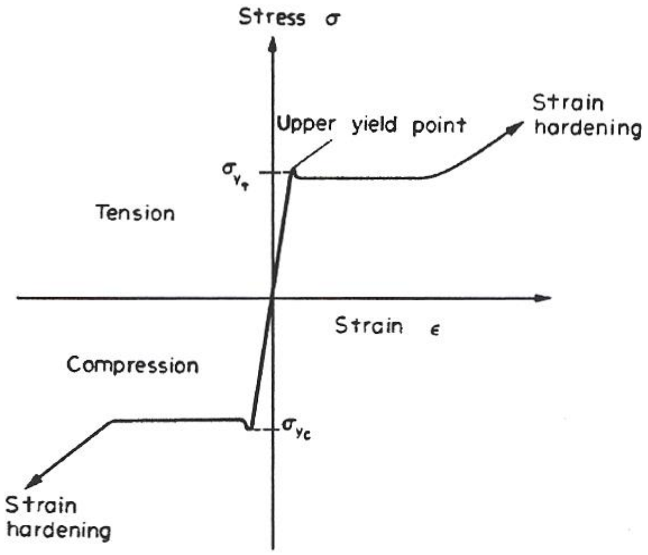

Fig. 11.1

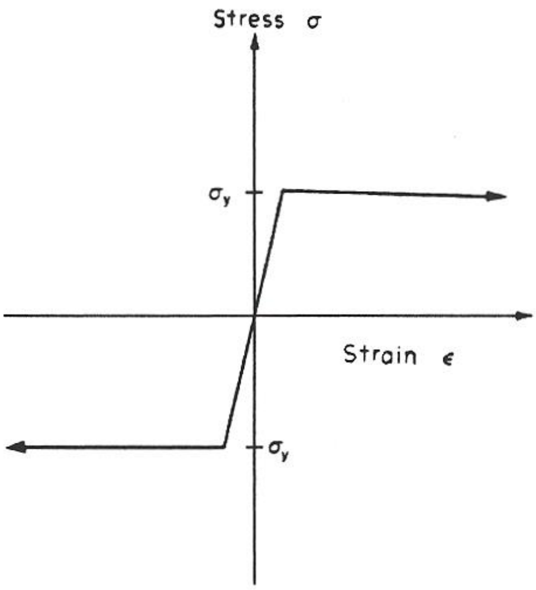

Fig.11.2

Figure 11.2 shows the assumed material behavior which:

(a) Ignores the presence of upper and lower yields and suggests only a single yield point;
(b) takes the yield stress in tension and compression to be equal;
(c) Assumes that yielding takes place at constant strain thereby ignoring any strain-hardening characteristics.
Thus, once the material has yielded, stress is assumed to remain constant throughout any further deformation.
It is further assumed, despite assumption (c), that transverse sections of beams in bending remain plane throughout the loading process, i.e. strain is proportional to distance from the neutral axis.
It is now possible on the basis of the above assumptions to determine the moment which must be applied to produce:

(a) the maximum or limiting elastic conditions in the beam material with yielding just initiated at the outer fibers;

(b) yielding to a specified depth;

(c) yielding across the complete section.

The latter situation is then termed a **fully plastic state, or "plastic hinge".** Depending on the support and loading conditions, one or more plastic hinges may be required before complete collapse of the beam or structure occurs, the load required to produce this situation then being termed the **collapse load.**

11.2. <u>Plastic bending of rectangular-sectioned beams</u>

Figure 11.3(a) shows a rectangular beam loaded until the yield stress has just been reached in the outer fibers. The beam is still completely elastic and the bending theory applies, i.e.

Maximum elastic moment

$$M = \frac{\sigma I}{y}$$

$\therefore$ $$\text{maximum elastic moment} = \sigma_y \times \frac{BD^3}{12} \times \frac{2}{D}$$

$$\boldsymbol{M_E = \frac{BD^2}{6}\sigma_y} \qquad \ldots(1)$$

If loading is then increased, it is assumed that instead of the stress at the outside increasing still further, more and more of the section reaches the yield stress σ_y. Consider the stage shown in Fig. 11.3(b).

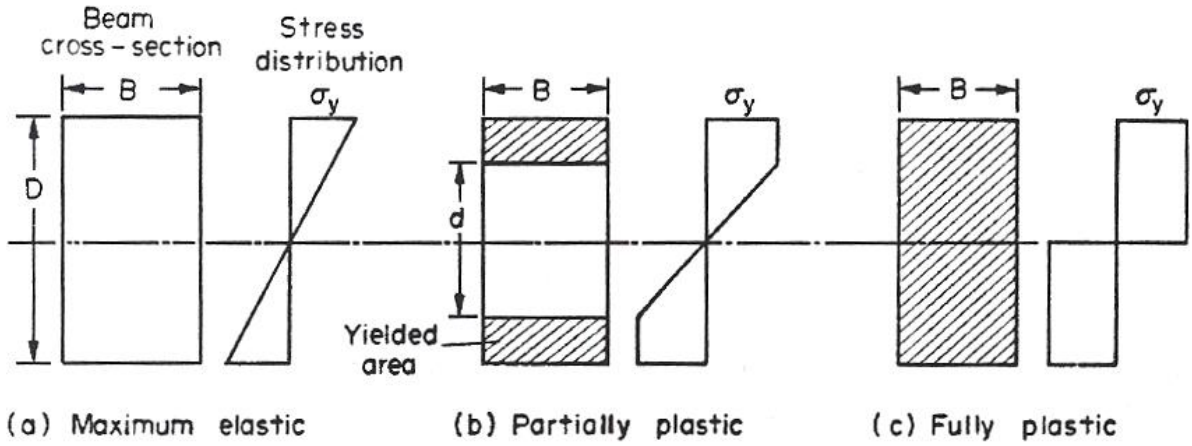

Fig.11.3. Plastic bending of rectangular-section beam.

Partially plastic moment,

$$M_{PP} = \text{moment of elastic portion} + \text{total moment of plastic portion}$$

$\therefore$ $$M_{PP} = \frac{Bd^2}{6}\sigma_y + 2\left\{\underbrace{\sigma_y}_{\text{stress}} \times \underbrace{B\left[\frac{D}{2} - \frac{d}{2}\right]}_{\text{area}} \underbrace{\left[\frac{1}{2}\left(\frac{D}{2} - \frac{d}{2}\right) + \frac{d}{2}\right]}_{\text{moment arm}}\right\}$$

$$M_{PP} = \sigma_y\left[\frac{Bd^2}{6} + \frac{B}{4}(D-d)(D+d)\right]$$

$$= \frac{B\sigma_y}{12}[2d^2 + 3(D^2 - d^2)] = \boldsymbol{\frac{B\sigma_y}{12}[3D^2 - d^2]} \qquad \ldots(2)$$

When loading has been continued until the stress distribution is as in Fig. 11.3(c) (assumed), the beam will collapse. The moment required to produce this fully plastic state can be obtained from eqn. (2), since d is then zero,

$$\text{fully plastic moment, } M_{FP} = \frac{B\sigma_y}{12} \times 3D^2 = \frac{\mathbf{BD^2}}{\mathbf{4}}\boldsymbol{\sigma_y} \qquad \text{...(3)}$$

where σ_y is the stress at the elastic limit, or yield stress.
This is the moment therefore which produces a plastic hinge in a rectangular-section beam.

11.3. <u>Shape factor - symmetrical sections</u>

The shape factor is defined as the ratio of the moments required to produce fully plastic and maximum elastic states:

$$\text{Shape factor } \lambda = \frac{\text{fully plastic moment}}{\text{maximum elastic moment}}$$

$$\textbf{shape factor } \boldsymbol{\lambda} = \frac{\mathbf{M_{FP}}}{\mathbf{M_E}} \qquad \text{...(4)}$$

It is a factor which gives a measure of the increase in strength or load-carrying capacity which is available beyond the normal elastic design limits for various shapes of section, e.g. for the rectangular section above,

$$\text{shape factor } = \frac{BD^2}{4}\sigma_y \bigg/ \frac{BD^2}{6}\sigma_y = \mathbf{1.5}$$

11.4. <u>Application to I-section beams</u>

When the B.M. applied to an I – section beam is just sufficient to initiate yielding in the extreme fiber, the stress distribution is as shown in Fig. 11.4(a) and the value of the moment is obtained from the simple bending theory by subtraction of values for convenient rectangles.

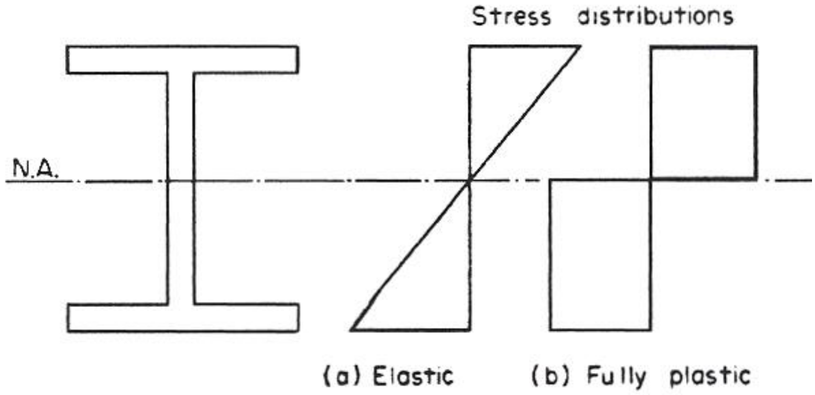

Fig. 11.4.Plastic bending of symmetrical (I-section) beam.

i.e.

$$M_E = \frac{\sigma I}{y}$$

$$= \sigma_y \left[\frac{BD^3}{12} - \frac{bd^3}{12}\right]\frac{2}{D}$$

If the moment is then increased to produce full plasticity across the section, i.e. a plastic hinge, the stress distribution is as shown in Fig. 11.4(b) and the value of the moment is obtained by applying eqn. (3) to the same convenient rectangles considered above.

$$M_{FP} = \sigma_y \left[\frac{BD^2}{4} - \frac{bd^2}{4}\right]$$

The value of the shape factor can then be obtained as the ratio of the above equations M_{FP}/M_E. A typical value of shape factor for commercial rolled steel joists is 1.18, thus indicating only an 18% increase in "strength" capacity using plastic design procedures compared with the 50% of the simple rectangular section.

11.5. Partially plastic bending of unsymmetrical sections

Consider the T-section beam shown in Fig. 11.5. Whilst stresses remain within the elastic limit the position of the N.A. can be obtained in the usual way by taking moments of area about some convenient axis. A typical position of the elastic N.A. is shown in the figure. Application of the simple bending theory about the N.A. will then yield the value of M_E as described in the previous paragraph.

Whatever the state of the section, be it elastic, partially plastic or fully plastic,

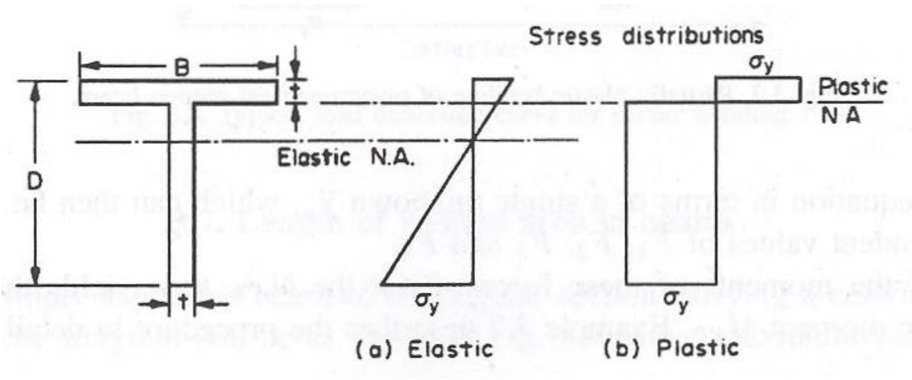

Fig. 11.5. Plastic bending of unsymmetrical (T-section) beam

equilibrium of forces must always be maintained, i.e. at any section the tensile forces on one side of the N.A. must equal the compressive forces on the other side.

∑stress x area above N.A. = **∑stress x area below NA.**

In the fully plastic condition, therefore, when the stress is equal throughout the section, the above equation reduces to

∑areas above N.A. = **∑areas below N.A.**...(5)

In the ultimate stage when a plastic hinge has been formed the N.A. will be positioned such that eqn. (5) applies, or, often more conveniently,

$$\textbf{area above or below N.A.} = \tfrac{1}{2}\ \textbf{total area} \qquad ...(6)$$

In the partially plastic state, as shown in Fig. 11.6, the N.A. position is again determined by applying equilibrium conditions to the forces above and below the N.A. The section is divided into convenient parts, each subjected to a force = average stress x area, as indicated, then

$$\mathbf{F_1+F_2 = F_3+F_4} \qquad ...(7)$$

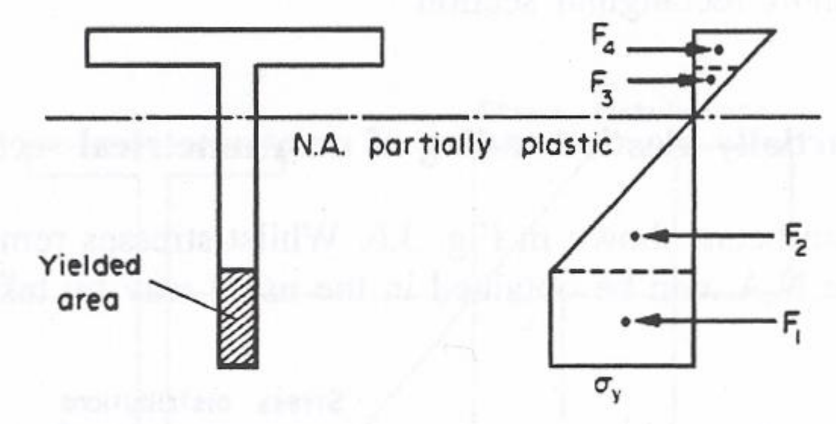

Fig. 11.6.Partially plastic bending of unsymmetrical section beam

and this is an equation in terms of a single unknown $\overline{y_p}$, which can then be determined, as can the independent values of F_1, F_2, F_3 and F_4.
The sum of the moments of these forces about the NA. then yields the value of the partially plastic moment.

11.6. <u>Deflections of partially plastic beams</u>

Deflections of partially plastic beams are calculated on the basis of the elastic areas only. In plastic limit or ultimate collapse load procedures the normal elastic safety factor is replaced by a load factor as follows:

$$\text{load factor} = \frac{\text{collapse load}}{\text{allowable working load}}$$

<u>Torques for plastic torsion</u>

For **solid shafts**, radius R, strained up to and beyond the elastic limit in shear, i.e. for plastic torsion, the torques which can be transmitted at each stage are

Maximum elastic torque ,
$$T_E = \frac{\pi R^3}{2}\tau_y$$

Partially plastic torque,
$$T_{PP} = \frac{\pi\tau_y}{6}[4R^3 - R_1^3] \quad \text{(yielding to radius } R_1\text{)}$$

Fully plastic torque,
$$T_{FP} = \frac{2\pi R^3}{3}\tau_y$$

Where τ_y is the shear stress at the elastic limit, or shear yield stress. Angles of twist of partially plastic shafts are calculated on the basis of the elastic core only.

For **hollow shafts**, inside radius R_1, outside radius R yielded to radius R_2,

$$T_{PP} = \frac{\pi\tau_y}{6R_2}[4R^3R_2 - R_2^4 - 3R_1^4]$$

$$T_{FP} = \frac{2\pi\tau_y}{3}[R^3 - R_1^3]$$

Plastic moment

For **eccentric loading** of rectangular sections the fully plastic moment is given by

$$M_{FP} = \frac{BD^2}{4}\sigma_y - \frac{P^2N^2}{4B\sigma_y}$$

where P is the axial load, N the load factor and B the width of the cross-section. The maximum allowable moment is then given by

$$M = \frac{BD^2}{4N}\sigma_y - \frac{P^2N}{4B\sigma_y}$$

Collapse speed

For a **solid rotating disc**, radius R, the collapse speed ω_p, is given by

$$\omega_p^2 = \frac{3\sigma_y}{\rho R^2}$$

where ρ is the density of the disc material.

For **rotating hollow discs,**the collapse speed is found from

$$\omega_p^2 = \frac{3\sigma_y}{\rho}\left[\frac{R_2 - R_1}{R_2^3 - R_1^3}\right]$$

EXAMPLES

1. (a) A rectangular-section steel beam, 50 mm wide by 20 mm deep, is used as a simply supported beam over a span of 2 m with the 20 mm dimension vertical. Determine the value of the central concentrated load which will produce initiation of yield at the outer fibers of the beam.

(b) If the central load is then increased by 10% find the depth to which yielding will take place at the centre of the beam span.

(c) Over what length of beam will yielding then have taken place?

(d) What are the maximum deflections for each load case?

For steel σ_y in simple tension and compression = 225 MN/m^2 and E = 206.8 GN/m^2.

Solution

(a) From eqn. (1) the B.M. required to initiate yielding is

$$\frac{BD^2}{6}\sigma_y = \frac{50 \times 20^2 \times 10^{-9}}{6} \times 225 \times 10^6 = 750 \text{ N m}$$

But the maximum B.M. on a beam with a central point load is WL/4, at the centre.

$$\therefore \quad \frac{W \times 2}{4} = 750$$

i.e. W = 1500 N

The load required to initiate yielding is 1500 N.

(b) If the load is increased by 10% the new load is

$$W' = 1500 + 150 = 1650 \text{ N}$$

The maximum B.M. is therefore increased to

$$M' = \frac{W'L}{4} = \frac{1650 \times 2}{4} = 825 \text{ Nm}$$

and this is sufficient to produce yielding to a depth d, and from eqn. (2),

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2] = 825 \text{ Nm}$$

$$\therefore \quad 825 = \frac{50 \times 10^{-3} \times 225 \times 10^6}{12}[3 \times 2^2 - d^2]10^{-4}$$

where d is the depth of the elastic core in centimetres,

$$\therefore \quad 8.8 = 12 - d^2$$

$$d^2 = 3.2 \text{ and } d = 1.79 \text{ cm}$$

$$\therefore \quad \text{depth of yielding} = \tfrac{1}{2}(D - d) = \tfrac{1}{2}(20 - 17.9) = \mathbf{1.05\ mm}$$

(c) With the central load at 1650 N the yielding will have spread from the centre as shown in Fig.7.7. At the extremity of the yielded region, a distance x from each end of the beam, the section will just have yielded at the extreme surface fibers, i.e. the moment carried at this section will be the maximum elastic moment and given by eqn. (1) - see part (a) above.

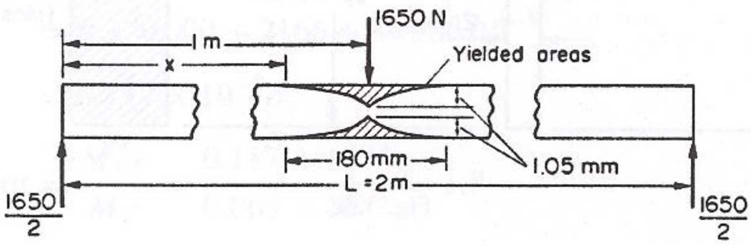

Fig. 11.7

Now the B.M. at the distance x from the support is

$$\frac{1650x}{2} = \frac{BD^2}{6}\sigma_y = 750$$

$$\therefore \quad x = \frac{2 \times 750}{1650} = 0.91 \text{ m}$$

Therefore length of beam over which yielding has occurred

= 2 - 2 x 0.91 = 0.18 m = 180 mm

(d) For W = 1500 N the beam is completely elastic and the maximum deflection, at the centre, is given by the standard form of equation:

$$\delta = \frac{WL^3}{48EI} = \frac{1500 \times 2^3 \times 12}{48 \times 206.8 \times 10^9 \times 50 \times 20^3 \times 10^{-12}}$$

$$= 0.0363 \text{ m} = \mathbf{36.3\ mm}$$

With W = 1650 N and the beam partially plastic, deflections are calculated on the basis of the elastic core only,

$$\text{i.e.} \qquad \delta = \frac{W'L^3}{48EI'} = \frac{1650 \times 2^3 \times 12}{48 \times 206.8 \times 10^9 \times 50 \times 17.9^3 \times 10^{-12}}$$
$$= 0.0556 \text{ m} = \mathbf{55.6 \text{ mm}}$$

2. (a) Determine the "shape factor" of a T section beam of dimensions 100 mm x 150 mm x 12 mm as shown in Fig.7.8.

(b) A cantilever is to be constructed from a beam with the above section and is designed to carry a uniformly distributed load over its complete length of 2 m. Determine the maximum u.d.I. that the cantilever can carry if yielding is permitted over the lower part of the web to a depth of 25 mm. The yield stress of the material of the cantilever is 225 MN/m^2.

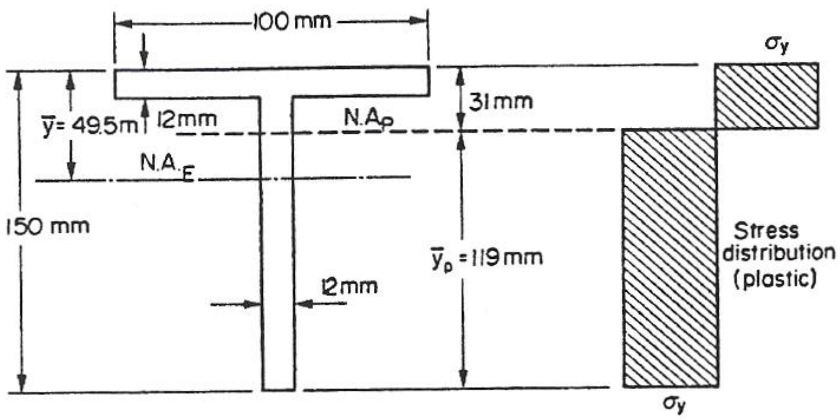

Fig. 11.8

Solution

(a)

$$\text{Shape factor} = \frac{\text{fully plastic moment}}{\text{maximum elastic moment}}$$

To determine the maximum moment carried by the beam while completely elastic we must first determine the position of the N.A.

Take moments of area about the top edge (see Fig.11.8):

$$(100 \times 12 \times 6) + (138 \times 12 \times 81) = [(100 \times 12) + (138 \times 12)]\ \bar{y}$$

$$7200 + 134136 = (1200 + 1656)\ \bar{y}$$

$$\therefore \qquad \bar{y} = 49.5 \text{ mm}$$

$$\therefore \qquad I_{NA} = \left[\frac{100 \times 49.5^3}{3} + \frac{12 \times 100.5^3}{3} - \frac{88 \times 37.5^3}{3}\right] 10^{-12} \text{ m}^4$$

$$= \tfrac{1}{3}[121.29 + 121.81 - 46.4]10^{-7}$$

$$= 6.56 \times 10^{-6} \text{ m}^4$$

Now from the simple bending theory the moment required to produce the yield stress at the edge of the section (in this case the lower edge), i.e. the maximum elastic moment, is

$$M_E = \frac{\sigma I}{y_{\max}} = \sigma_y \times \frac{6.56 \times 10^{-6}}{100.5 \times 10^{-3}} = 0.065 \times 10^{-3} \sigma_y$$

When the section becomes fully plastic the N.A. is positioned such that

area below NA. = half total area

i.e. if the plastic N.A. is a distance above the base, then

$$\bar{y}_p \times 12 = \tfrac{1}{2}(1200 + 1656)$$

$$\therefore \qquad \bar{y}_p = 119 \text{ mm}$$

The fully plastic moment is then obtained by considering the moments of forces on convenient rectangular parts of the section, each being subjected to a uniform stress σ_y,

$$\text{i.e.} \qquad M_{FP} = \left[\sigma_y(100 \times 12)(31 - 6) + \sigma_y(31 - 12) \times 12 \times \frac{(31 - 12)}{2} \right.$$

$$\left. + \sigma_y(119 \times 12)\frac{119}{2}\right] 10^{-9}$$

$$= \sigma_y(30\,000 + 2166 + 84\,966)10^{-9}$$

$$= 0.117 \times 10^{-3} \sigma_y$$

$$\therefore \qquad \text{shape factor} = \frac{M_{FP}}{M_E} = \frac{0.117 \times 10^{-3}}{0.065 \times 10^{-3}} = \mathbf{1.8}$$

(b) For this part of the question the load on the cantilever is such that yielding has progressed to a depth of 25 mm over the lower part of the web. It has been shown in section 7.5 that whilst plastic penetration proceeds, the N.A. of the section moves and is always positioned by the rule:

compressive force above N.A. = tensile force below N.A.

Thus if the partially plastic N.A. is positioned a distance y above the extremity of the yielded area as shown in Fig.11.9, the forces exerted on the various parts of the section may be established (proportions of the stress distribution diagram being used to determine the various values of stress noted in the figure).

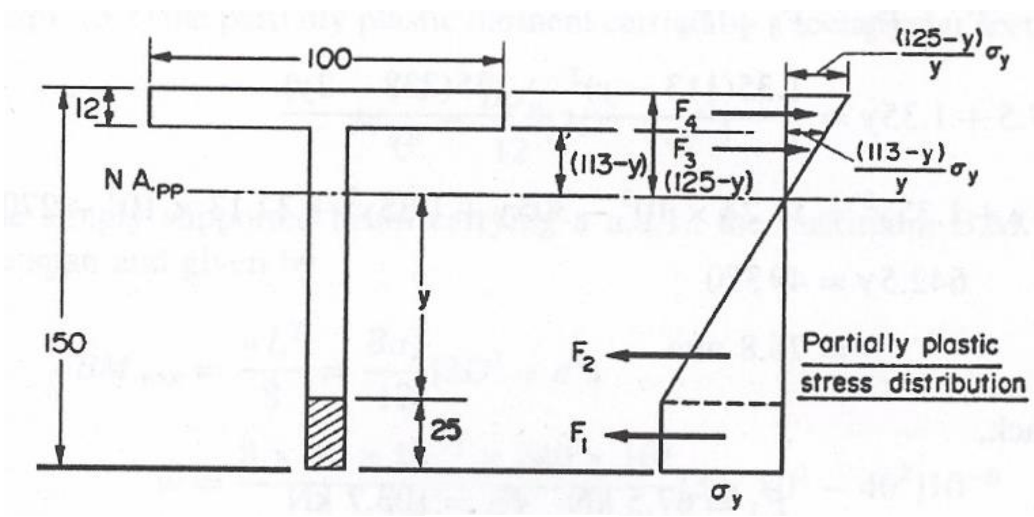

Fig. 11.9

Force on yielded area F_1 = stress x area

$$= 225 \times 10^6 (12 \times 25 \times 10^{-6})$$
$$= 67.5 \text{ kN}$$

Force on elastic portion of web below N.A.

$$F_2 = \text{ average stress } \times \text{ area}$$
$$= \frac{225 \times 10^6}{2} (12 \times y \times 10^{-6})$$
$$= 1.35y \text{ kN}$$

where y is in millimetres.

Force in web above N.A.

$$F_3 = \text{ average stress } \times \text{ area}$$
$$= \frac{(113-y)}{2y}(225 \times 10^6)(113-y)12 \times 10^{-6}$$
$$= 1.35\frac{(113-y)^2}{y} \text{ kN}$$

Force in flange

$$F_4 = \text{ average stress } \times \text{ area}$$
$$= \frac{1}{2}\left[\frac{(113-y)}{y} + \frac{(125-y)}{y}\right](225 \times 10^6)100 \times 12 \times 10^{-6} \text{ approximately}$$
$$= \frac{(238-2y)}{2y} 225 \times 10^6 \times 100 \times 12 \times 10^{-6}$$
$$= 135\frac{(238-2y)}{y} \text{ kN}$$

Now for the resultant force across the section to be zero,

$$F_1 + F_2 = F_3 + F_4$$
$$67.5 + 1.35y = \frac{1.35(113-y)^2}{y} + \frac{135(238-2y)}{y}$$
$$\therefore \quad 67.5y + 1.35y^2 = 17.24 \times 10^3 - 305y + 1.35y^2 + 32.13 \times 10^3 - 270y$$
$$642.5y = 49\,370$$
$$y = 76.8 \text{ mm}$$

Substituting back,

F_1= 67.5 kN F_2 = 103.7 kN

F_3 = 23 kN F_4 = 148.1 kN

The moment of resistance of the beam can now be obtained by taking the moments of these forces about the N.A. Here, for ease of calculation, it is assumed that F_4 acts at the mid-point of the web. This, in most cases, is sufficiently accurate for practical purposes.

$$\text{Moment of resistance} = \left\{F_1(y+12.5) + F_2\left(\frac{2y}{3}\right) + F_3\left[\frac{2}{3}(113-y)\right] + F_4[(113-y)+6]\right\} 10^{-3} \text{ kNm}$$
$$= (6030 + 5312 + 554 + 6243)10^{-3} \text{ kNm}$$
$$= 18.14 \text{ kNm}$$

Now the maximum B.M. present on a cantilever carrying a u.d.l. is $wL^2/2$ at the support

$$\therefore \qquad \frac{wL^2}{2} = 18.15 \times 10^3$$

The maximum u.d.l. which can be carried by the cantilever is then

$$w = \frac{18.15 \times 10^3 \times 2}{4} = \mathbf{9.1\ kN/m}$$

3.(a) A steel beam of rectangular section, 80 mm deep by 30 mm wide, is simply supported over a span of 1.4 m and carries a u.d.I. w. If the yield stress of the material is 240 MN/m^2, determine the value of w when yielding of the beam material has penetrated to a depth of 20 mm from each surface of the beam.

(b) What will be the magnitudes of the residual stresses which remain when load is removed?

(c) What external moment must be applied to the unloaded beam in order to return it to its undeformed (straight) position?

Solution

(a) From eqn. (2) the partially plastic moment carried by a rectangular section is given by

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2]$$

Thus, for the simply supported beam carrying a u.d.l., the maximum B.M. will be at the centre of the span and given by

$$BM_{\max} = \frac{wL^2}{8} = \frac{B\sigma_y}{12}[3D^2 - d^2]$$

$$\therefore \qquad w = \frac{8 \times 30 \times 10^{-3} \times 240 \times 10^6}{1.4^2 \times 12}[3 \times 80^2 - 40^2]10^{-6}$$

$$= \mathbf{43.1\ kN/m}$$

(b) From the above working

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2] = \frac{wL^2}{8}$$

$$= 43.1 \times 10^3 \times \frac{1.4^2}{8} = 10.6\ \text{kNm}$$

During the unloading process a moment of equal value but opposite sense is applied to the beam assuming it to be completely elastic. Thus the equivalent maximum elastic stress σ' introduced at the outside surfaces of the beam by virtue of the unloading is given by the simple bending theory with M = M_{pp}= 10.6 kNm,

i.e.

$$\sigma' = \frac{My}{I} = \frac{10.6 \times 10^3 \times 40 \times 10^{-3} \times 12}{30 \times 80^3 \times 10^{-12}}$$
$$= 0.33 \times 10^9 = 330 \text{ MN/m}^2$$

The unloading, elastic stress distribution is then linear from zero at the N.A. to ±330 MN/m^2 at the outside surfaces, and this may be subtracted from the partially plasticloading stress distribution to yield the residual stresses as shown in Fig. 11.10.

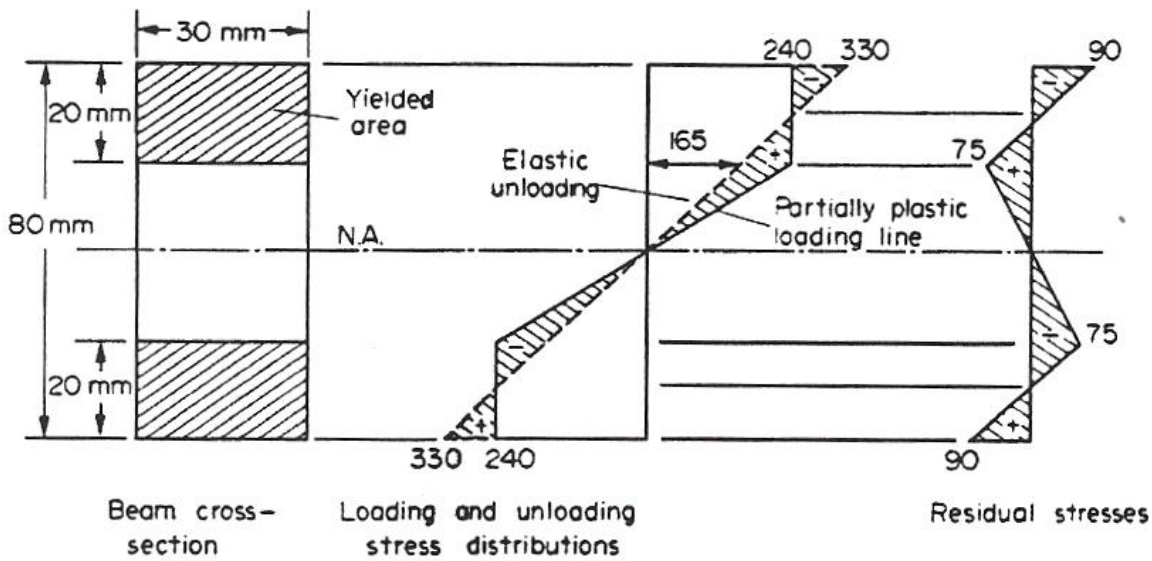

Fig.11.10

(c) The residual stress distribution of Fig. 11.10 indicates that the central portion of the beam, which remains elastic throughout the initial loading process, is subjected to a residual stress system when the beam is unloaded from the partially plastic state. The beam will therefore be in a deformed state. In order to remove this deformation an external moment must be applied of sufficient magnitude to return the elastic core to its unstressed state. The required moment must therefore introduce an elastic stress distribution producing stresses of ±75 MN/m^2 at distances of 20 mm from the N.A. Thus, applying the bending theory,

$$M = \frac{\sigma I}{y} = \frac{75 \times 10^6}{20 \times 10^{-3}} \times \frac{30 \times 80^3 \times 10^{-12}}{12}$$
$$= \mathbf{4.8\ kNm}$$

Alternatively, since a moment of 10.6 kNm produces a stress of 165 MN/m^2 at 20 mm from the N.A., then, by proportion, the required moment is

$$M = 10.6 \times \frac{75}{165} = \mathbf{4.8\ kNm}$$

4. A solid circular shaft, of diameter 50 mm and length 300 mm, is subjected to a graduallyincreasing torque T. The yield stress in shear for the shaft material is 120 MN/m^2 and, up to the yield point, the modulus of rigidity is 80 GNIm^2.

(a) Determine the value of T and the associated angle of twist when the shaft material first yields.

(b) If, after yielding, the stress is assumed to remain constant for any further increase in strain, determine the value of T when the angle of twist is increased to twice that at yield.

Solution

(a) For this part of the question the shaft is elastic and the simple torsion theory applies,

i.e. $$T = \frac{\tau J}{R} = \frac{120 \times 10^6}{25 \times 10^{-3}} \times \frac{\pi(25 \times 10^{-3})^4}{2} = 2950$$
$$= \mathbf{2.95\ kNm}$$
$$\theta = \frac{\tau L}{GR} = \frac{120 \times 10^6 \times 300 \times 10^{-3}}{80 \times 10^9 \times 25 \times 10^{-3}} = 0.018 \text{ radian}$$
$$= \mathbf{1.03^\circ}$$

If the torque is now increased to double the angle of twist the shaft will yield to some radius R_1. Applying the torsion theory to the elastic core only,

$$\theta = \frac{\tau L}{GR}$$

i.e. $$2 \times 0.018 = \frac{120 \times 10^6 \times 300 \times 10^{-3}}{80 \times 10^9 \times R_1}$$

$\therefore$ $$R_1 = \frac{120 \times 10^6 \times 300 \times 10^{-3}}{2 \times 0.018 \times 80 \times 10^9} = 0.0125 = 12.5 \text{ mm}$$

Therefore partially plastic torque is given by:

$$= \frac{\pi\tau_y}{6}[4R^3 - R_1^3]$$

$$= \frac{\pi \times 120 \times 10^6}{6}[4 \times 25^3 - 12.5^3]10^{-9}$$

$$= \mathbf{3.8\ kNm}$$

5.A 50 mm diameter steel shaft is case-hardened to a depth of 2 mm. Assuming that theinner core remains elastic up to a yield stress in shear of 180 MN/m^2 and that the case can also be assumed to remain elastic up to failure at the shear stress of 320 MN/m^2, calculate:

(a) the torque required to initiate yielding at the outside surface of the case; (b) the angle of twist per meter length at this stage.

Take G = 85 GN/m^2 for both case and core whilst they remain elastic.

Solution

Since the modulus of rigidity G is assumed to be constant throughout the shaft whilst elastic, the angle of twist θ will be constant.

The stress distribution throughout the shaft cross-section at the instant of yielding of the outside surface of the case is then as shown in Fig. 7.11, and it is evident that whilst the failure stress of 320 MN/m^2 has only just been reached at the outside of the case, the yield stress of the core of 180 NM/m^2 has been exceeded beyond a radius r producing a fully plastic annulus and an elastic core.

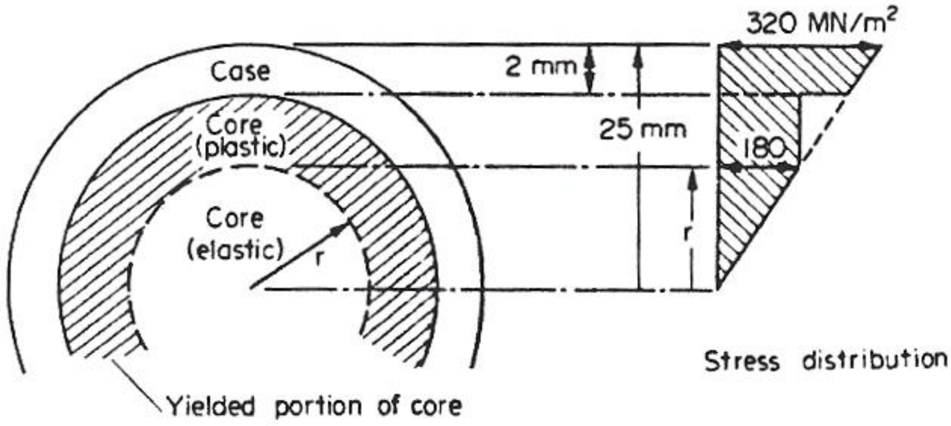

Fig. 11.11

By proportions, since $G_{case} = G_{core}$, then

$$\left(\frac{\tau}{r}\right)_{case} = \left(\frac{\tau}{r}\right)_{core}$$

$$\frac{180}{r} = \frac{320}{25}$$

$$\therefore \quad r = \frac{180}{320} \times 25 = 14.1 \text{ mm}$$

The shaft can now be considered in three parts:

(i) A solid elastic core of 14.1 mm external radius;

(ii) A fully plastic cylindrical region between r = 14.1 mm and r = 23 mm;

(iii) An elastic outer cylinder of external diameter 50 mm and thickness 2 mm.

$$\text{Torque on elastic core} = \frac{\tau_y J}{R} = \frac{180 \times 10^6}{14.1 \times 10^{-3}} \times \frac{\pi(14.1 \times 10^{-3})^4}{2}$$

$$= 793 \text{ Nm } = 0.793 \text{ kNm}$$

$$\text{Torque on plastic section} = 2\pi\tau_y \int_{r_1}^{r_2} r^2\, dr$$

$$= \frac{2\pi \times 180 \times 10^6}{3}[23^3 - 14.1^3]10^{-9}$$

$$= \frac{2\pi \times 180 \times 10^6 \times 9364 \times 10^{-9}}{3}$$

$$= 3.53 \text{ kNm}$$

$$\text{Torque on elastic outer case} = \frac{\tau_y J}{r} = \frac{320 \times 10^6}{25 \times 10^{-3}}\pi \left[\frac{25^4 - 23^4}{2}\right] 10^{-12}$$

$$= 2.23 \text{ kNm}$$

Therefore total torque required = $(0.793 + 3.53 + 2.23)10^3$

= 6.55 kNm

Since the angle of twist is assumed constant across the whole shaft its value may be determined by application of the simple torsion theory to either the case or the elastic core.

$$\text{For the case:} \quad \frac{\theta}{L} = \frac{\tau}{GR} = \frac{320 \times 10^6}{85 \times 10^9 \times 25 \times 10^{-3}}$$

$$= 0.15 \text{ rad} = \mathbf{8.6^\circ}$$

6.A hollow circular bar of 100 mm external diameter and 80 mm internal diameter (Fig. 7.12) is subjected to a gradually increasing torque T. Determine the value of T:

(a) when the material of the bar first yields;

(b) when plastic penetration has occurred to a depth of 5 mm;

(c) when the section is fully plastic.

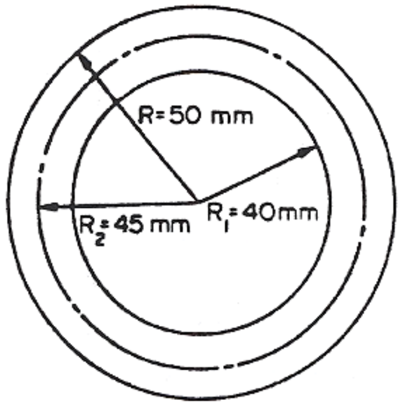

Fig. 11.12

The yield stress in shear of the shaft material is 120 MN/m^2.

Determine the distribution of the residual stresses present in the shaft when unloaded from conditions (b) and (c).

<u>Solution</u>(a) Maximum elastic torque is given by:

$$= \frac{\pi\tau_y}{2R}[R^4 - R_1^4] = \frac{\pi \times 120 \times 10^6}{2 \times 50 \times 10^{-3}}(625 - 256)10^{-8}$$
$$= 13900 \text{ Nm} = \mathbf{13.9 \text{ kNm}}$$

(b) Partially plastic torque is given by:

$$= \frac{\pi\tau_y}{2R_2}[R_2^4 - R_1^4] + \frac{2\pi\tau_y}{3}[R^3 - R_2^3]$$
$$= \frac{\pi \times 120 \times 10^6}{2 \times 45 \times 10^{-3}}(4.5^4 - 256)10^{-8} + \frac{2\pi \times 120 \times 10^6}{3}(125 - 91)10^{-6}$$
$$= 6450 + 8550 = 15000 \text{ Nm} = \mathbf{15 \text{ kNm}}$$

(c) Fully plastic torque is given by:

$$= \frac{2\pi\tau_y}{3}[R^3 - R_1^3]$$
$$= \frac{2\pi \times 120 \times 10^6}{3}[125 - 64]10^{-6} = 15330 = \mathbf{15.33 \text{ kNm}}$$

The unloading stress distribution is then linear, from zero at the centre of the bar to 129 MN/m^2 at the outside. This can be subtracted from the partially plastic loading stress distribution as shown in Fig. 7.13 to produce the residual stress distribution shown.

Similarly, unloading from the fully plastic state is equivalent to applying an elastic torque of 15.33 kNm of opposite sense. By proportion, from the above calculation,

Subtracting the resulting unloading distribution from the fully plastic loading one gives the

$$\text{equivalent stress at outside of shaft on unloading} = \frac{15.33}{15} \times 129 = 132 \text{ MN/m}^2$$

residual stresses shown in Fig. 11.14.

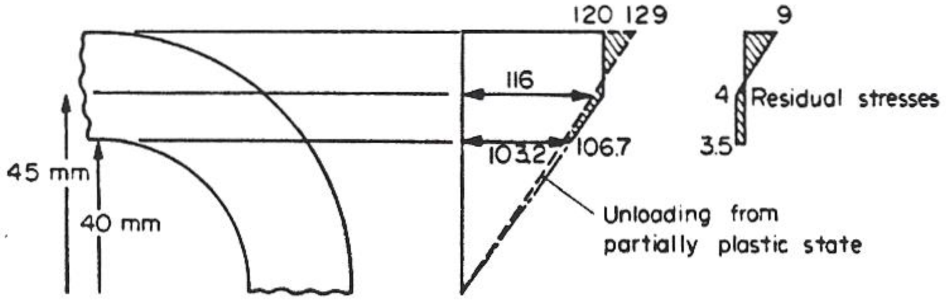

Fig. 7.13

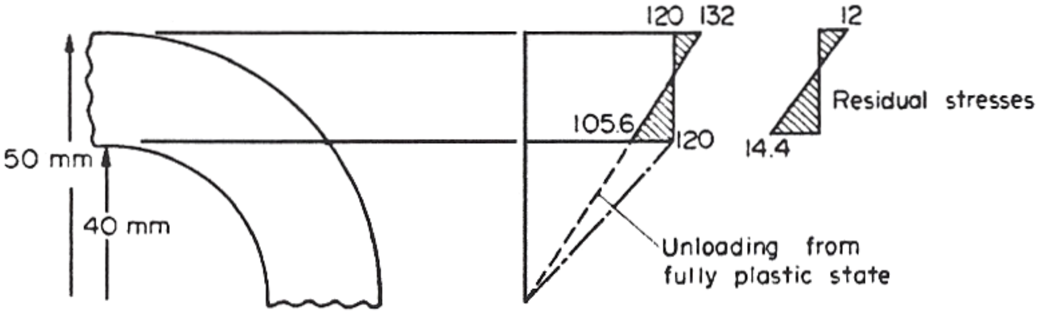

Fig.7.14